Symmetric Spaces of Exceptional Groups

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Abstract

We adress the problem of the reasons for the existence of 12 symmetric spaces with the exceptional Lie groups. The $1+2$ cases for G_2 and F_4 respectively are easily explained from the octonionic nature of these groups. The $4+3+2$ cases on the $E_{6,7,8}$ series require the magic square of Freudenthal and, for the split case, an appeal to the supergravity chain in $5, 4$ and 3 spacetime dimensions.

1 Introduction

Symmetric Spaces were introduced and classified by E. Cartan in 1926. They are riemannian spaces with an extra symmetry condition, namely geodesic reflection being isometry; they turn out to be homogeneous spaces $S = G/K$. This has as consequence that curvature is covariant constant, $\nabla R = 0$; that fact can be used as an alternative definition. Here we shall refer mainly to the *compact* subclass; the standard source is the book by Helgason [1]. From the Lie algebra $L = L(G)$ point of view, there is an involutory automorphism with $L(K)$ as the fix point set. Then if P is a suplementary space, so $L(G) = L(K) \dot{+} P$, $[L(K), L(K)] = L(K)$, $[L(K), P] = P$, and $[P, P] = L(K)$. A good recent reference for physicists in this context is [2]. If G and K are compact, with Weyl´s unitary trick $P \to iP$ we obtain the *noncompact* families of symmetric spaces.

There are *seven families* of symmetric spaces associated to the *three division algebras*, the reals R, the complex C and the quaternions H, as the transitive groups (G above) in general are $O(n), U(n)$ and $Sq(n)$, and there are also *twelve single* cases. The aim of this work is to understand the 12 exceptions as extensions of the classical families to the

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octonion algebra O.

This study must be seen as a case of understanding exceptional cases in mathematics: once one has classified objects, as e.g. in regular and exceptional classes, the question arises to place exceptions in some context, to see why they are there. One can ask, for example, why there are exceptional Lie groups (like E_8), or sporadic (finite simple) groups (like the Monster group), or why some regular polytopes exist in four dimensions only (like the $24-$cell). Indeed, our problem has relation with other problems in Lie group theory: what are the *real forms* of (complex) Lie groups (classical or exceptional), which groups act *transitively on spheres*, and also with Berger´s classification of the *special holonomy* manifolds; in this late case, for example, there are, besides the $3 \cdot 2 = 6$ families, for the groups $SO, O;\ SU, U;\ Sq$ and $q(n) := [Sq(n) \times Sq(1)]/Z_2$, also two exceptions, again related to octonions, namely G_2 and Spin(7).

It is of course the lack of associativity of the octonion division algebra which makes the corresponding "series" finite in number.

So we are adressing a purely mathematical question. We hope, however, that the issue would be relevant for physics: exceptional groups appear conspicuously e.g. in string or M-Theory [3], and on general grounds one expects Nature to be described by a singular, non-generic mathematical structure, as Nature is unique (!). There are also recent applications of symmetric spaces to integrable systems and to random matrices. We shall make use of Supergravity in relation with symmetric spaces at the end of this report.

2 Classification of Compact Symmetric Spaces

The conventional (generic) symmetric spaces are very easy to describe. Let us write the series of groups and inclusions for the Orthogonal (O), Unitary (U) and compact Symplectic (Sq) series:

$$O(n) \subset U(n) \subset Sq(n) \subset U(2n = m) \subset O(2m) \tag{1}$$

Roughly speaking, the four first classes of generic compact riemannian symmetric spaces are U/O, Sq/U, $U(2)/Sq$ and $O(2)/U$.

NOTE: For us $Sq(n)$ is the *compact* form of the C_n series of Cartan´s list of simple Lie groups; we shall use also (as above) the notation $q(n)$ for the "nonunimodular" form of $Sq(n)$. These should be understood as the "unitary" groups with quaternion entries, or with (double) unitary complex entries fixing an (extra) conjugation; "q" is for quaternion, "Sq" for unimodular quaternion, of course.

The first four uniparametric families of (compact) symmetric spaces are just the consecutive quotients above:

$$SU(n)/SO(n), \quad Sq(n)/U(n), \quad SU(2n)/Sq(n), \quad \text{and} \quad SO(2n)/U(n). \tag{2}$$

$$Dim \quad (n-1)(n+2)/2, \quad n(n+1), \quad (2n+1)(n-1), \text{ and } n(n-1). \tag{3}$$

$$Cartan\ notation : AI, \quad CI, \quad AII, \text{ and } DIII. \tag{4}$$

Besides, there are three biparametric families, with the same field, giving the grassmannians or collection of subspaces, namely $K(p+q)/K(p) \times K(q)$, where $K := R, C, H$ or groups O, U and Sq:

$$SO(p+q)/SO(p) \times SO(q), \quad SU(p+q)/SU(p) \times SU(q), \quad Sq(p+q)/Sq(p) \times Sq(q). \tag{5}$$

with dimensions pq, $2pq$ and $4pq$ respectively. In Cartan´s notation they are BDI, $AIII$ and CII.

These grasmannians are, as said, collection of subspaces; in particular, the important class of projective spaces are reported here, as e.g.

$$RP^n = O(n+1)/O(n) \times O(1) = SO(n+1)/O(n) = S^n/S^0 \tag{6}$$

Similarly

$$CP^n = ... = SU(n+1)/U(n) = S^{2n-1}/S^1, \quad HP^n = ... = Sq(n+1)/q(n) = S^{4n-1}/S^3 \tag{7}$$

The Lie groups themselves are also symmetric spaces, because one can write $G = [G \times G]/G_{diag}$. But we do not consider them further.

Symmetric spaces have also a *rank*, the dimension of the maximal totally geodesic flat submanifold. It corresponds to the abelian (Cartan) subalgebra of the Lie algebra of simple Lie groups. Symmetric spaces of rank one are, for example, the projective spaces and the spheres, e.g. $S^n = SO(n+1)/SO(n)$ and $S^{2n-1} = SU(n)/SU(n-1)$, as $SO(1) = SU(1) = 1$. These spaces are also called *two-point homogeneous spaces*, Cfr. [1]. The free classical motion on them is superintegrable, maximally antiergodic (the orbits filling up just a curve (or $1d$ manifold)), and quantization yields a discrete spectrum, depending on a single quantum number.

There remain the twelve exceptional symmetric spaces, $S = G/H$, where G is one of the five exceptional groups. Cartan found the $1+2+4+3+2 = 12$ symmetric spaces associated respectively to the exceptional groups G_2, F_4, E_6, E_7 and E_8. To understand them, firts we have to see precisely the relation they have with the octonion division algebra.

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3   Generalities on Octonions and Related Groups

We start from the reals R; it is an ordered field, $x < y$ makes sense, and there are no automorphims, Aut(R)=1. Now in R^2 we define an $R-$algebra by $i^2 = -1$, where $i = (0, 1)$. Then the complex numbers C are $z = x + iy$ (Gauss). Conjugation, norm and inverse are given as $\bar{z} = x - iy$, $N(z) = \bar{z}z$, and for $z \neq 0$, $z^{-1} = \frac{\bar{z}}{N(z)}$. Conjugation is the only continuous automorphism, $\text{Aut}_c(C) = Z_2$; the field C is no longer ordered; both R and C are *composition algebras*, meaning the norm verifies $N(z_1 z_2) = N(z_1)N(z_2)$. We have the inclusions $R^2 \supset S^1 \supset S^0$ for all complex numbers C, the unit complex numbers, and the imaginary units $\pm i$.

For the *quaternions H* we proceed similarly (Hamilton, 1842). In R^4 define $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$ and $k = ij = (0, 0, 0, 1)$ with $i^2 = j^2 = k^2 = -1$, which requires $ij + ji = 0$. Conjugation, norm and inverse are defined likewise as $q := u + ix + jy + kz \equiv Re(q) + Im(q)$ and then $\bar{q} = Re(q) - Im(q)$, $N(q) = \bar{q}q$, and for $q \neq 0$, $q^{-1} = \frac{\bar{q}}{N(q)}$. The inclusions now are $R^4 \supset S^3 \supset S^2$ for all quaternions H, the unit quaternions and the imaginary units. Conjugation now is *antiautomorphism*, but arbitrary rotations of the imaginary units are the automorphims, $\text{Aut}(H) = SO(3)$. H is an anticommutative (skew-)field. Note $N(q_1 q_2) = \overline{q_1 q_2}\, q_1 q_2 = \bar{q_2}\bar{q_1} q_1 q_2 = N(q_1)N(q_2)$

For the *octonions O* we proceed similarly (Greaves, 1842; Cayley, 1845). In R^8 define three independent antiinvolutory anticommutative units i, j, k (e.g. with $i^2 = j^2 = k^2 = -1$, $ij + ji = 0$ etc). Now *force* the same properties for the products $ij, jk, ki; (ij)k$: this implies nonassociativity, namely $(ij)k = -i(jk)$, the most remarkable (for unusual) property of octonions. There are $1 + \binom{3}{1} + \binom{3}{2} + 1 = 8$ units, seven imaginary. Conjugation, norm and inverse are defined likewise; the automorphim group should be a subgroup of $SO(7)$: the inclusions now are $R^8 \supset S^7 \supset S^6$. So Aut($O$) is G_2, the first (smallest) exceptional simple Lie group, of dimension 14 and rank 2. The octonions form a composition division noncommutative nonassociative algebra; in particular $N(o_1 o_2) = N(o_1)N(o_2)$ still holds. The concrete expression $(ij)k = -i(jk)$ is called *alternativity*, and does characterize the octonions: in general $[o_1 o_2 o_3] := (o_1 o_2)o_3 - o_1(o_2 o_3)$ is fully antisymmetric. For a further study, consult [4].

For the reals R we have the units $S^0 = (\pm 1) = Z_2 = O(1)$. For the complex C the units are $S^1 = \exp(i\phi) = SO(2) = U(1)$. For the quaternions H we have $S^3 = \text{Spin}(3) = \text{SU}(2) = \text{Sq}(1)$. But for the octonions O the sphere S^7 is *not* a group, only a parallelizable manifold (i.e. the tangent bundle is trivial). However, as shown by Ramond [5], the seven sphere becomes a group after stabilization by $G_2 = \text{Aut}(O)$, and becomes the group Spin(7)!

$$\text{Spin}(7) \sim \text{G}_2 \vee \text{S}^7 \tag{8}$$

We need now some results from Lie group theory. Firts, any compact simple Lie group can be understood (Hopf, ca. 1940) as finitely twisted product of odd-dimensional spheres,

symbolically

$$G \sim \Pi_j \odot S^{2j+1} \tag{9}$$

where j are called *the exponents*; for all this see [6]. For example. $SU(2) = S^3$ is known, $\mathrm{Spin}(4) = S^3 \times S^3$ is true, but also $SU(3) \sim S^3 \odot S^5$, etc. Other result is that the exponents are palindromic (Kostant), i.e., the differences left-right are the same. For example, the exponents j for $\mathrm{Spin}(10)$ are $(1, 3, 4, 5, 7)$.

Now we compare $Sq(1) = S^3$ and $q(1) = S^3 \times_2 S^3$ with $G_2 \sim S^3 \times S^{11}$ and $\mathrm{Spin}(7) \sim S^3 \times S^7 \times S^{11}$. We shall call

$$S^7 \sim `Oct(1)', \quad G_2 \sim SOct(1), \quad \mathrm{Spin}(7) \sim Oct(1) \tag{10}$$

on the understanding that $`Oct(1)'$ is NOT a group. To justify the notation, we remark: $Sq(1) \rightarrow q(1) \rightarrow S^3$ corresponds to $SOct(1) \rightarrow Oct(1) \rightarrow S^7$ which is true: $Spin(7)/G_2 = S^7$. There are more examples, e.g. $Sq(1)^2 = S^3 \times S^3$ corresponds to $Oct(1)^2 = S^3 \odot S^7 \odot S^7 \odot S^{11} = \mathrm{Spin}(8)$. Now as $q(2) \sim Sq(1) \times Sq(2) \sim S^3 \odot S^3 \odot S^7$, it leads to $Oct(2) \sim S^3 \odot S^7 \odot S^{11} \odot S^{15} = \mathrm{Spin}(9)$!

We can go up to $n = 3$, but there is a surprise! $`Oct(3)'$ would have exponents $j : (1, 3, 5, 7, 11)$, as corresponds to spheres $(7, 15, 23)$ stabilized by $(3, 11)$. But this combination is NOT palindromic! However, it becomes so by losing the S^7 sphere $((1,5,7,11)$ is OK!). Losing the 7-sphere is the unimodular (”S”) restriction, and indeed we have concocted F_4:

$$F_4 \sim S^3 \odot S^{11} \odot S^{15} \odot S^{23} = SOct(3) \tag{11}$$

which is the second exceptional Lie group, with dimension $52 = 3 + 11 + 15 + 23$ and rank 4 (number of spheres)!

We stop here for the moment. As summary, we have "shown" the following "equivalences" $`Oct(1)' = S^7$, $G_2 = \mathrm{Aut}(O) \sim S^3 \odot S^{11} = SOct(1)$, $Oct(1) = S^3 \odot S^7 \odot S^{11} = \mathrm{Spin}(7)$, $Oct(1)^2 = \mathrm{Spin}(8)$, $Oct(2) = \mathrm{Spin}(9)$, $SOct(3) = F_4$.

4   Symmetric Spaces of Exceptional Groups: G_2 and F_4

We are ready now no address the exceptional symmetric spaces. As we say that $G_2 = SOct(1)$ and $\mathrm{Spin}(7) = Oct(1)$, we generalize the $Sq(1)$ case: its only symmetric space is of q/U type, namely $Sq(1)/U(1) \approx S^3/S^1 = S^2 = CP^1$. Hence the octonionic equivalent must be

$$SOct(1)/q(1) \approx G_2/SO(4) \approx \frac{S^3 \odot S^{11}}{S^3 \odot S^3} \approx HP^2 \qquad (12)$$

It is the *only* symmetric space with G_2. We do not care about the "Octonionic" forms for Spin(7, 8, 9), so our next case will be F_4. Recalling $F_4 = SOct(3)$, there are now two possibilities: Oct/Sq and $Oct(3)/Oct(1) \times Oct(2)$; both are present: from

$$O(2) \subset U(2) \subset q(2) \subset Oct(2) = \text{Spin}(9) \qquad (13)$$

$$O(3) \subset U(3) \subset q(3) \subset SOct(3) = F_4 \qquad (14)$$

we get $SOct(3)/Oct(2) = F_4/\text{Spin}(9) = OP^2$, i.e., the Moufang (octonionic, projective) plane. But also $SOct(3)/q(3) = F_4/Sq(3) \times Sq(1)$, the other symmetric F_4-space.

There is another case, namely $Oct(2)/Oct(1)^2 \approx \text{Spin}(9)/\text{Spin}(8) = S^8 = OP^1$, but this is already accounted for, e.g. as $S^8 = SO(9)/SO(8)$.

Are there more cases? One proves that all other putative cases have been already accounted for. For example, from $U(2)/U(1) = S^3$ and $q(2)/q(1) = S^7$, one would expect $Oct(2)/Oct(1) = S^{15}$. Indeed $\text{Spin}(9)/\text{Spin}(7) = SO(9)/SO(7) \equiv SO(9)/SO(7) \times SO(1)$, *dejà vu*, as it is an orthogonal grassmannian (and S^{15} plays the role of a Stiefel manifold); it corresponds to the principal bundle $\text{Spin}(7) \to \text{Spin}(9) \to S^{15}$, an exceptional transitive action on spheres: in fact, $\text{Spin}(9)$ is the largest Spin group acting *trans* in any maximal sphere!

So we have $S = G_2/SO(4)$ with dim $S = 8$; it is Cartan´s G space. And $S' = F_4/\text{Spin}(9)$ with dim 16; it is Cartan´s FII space. Finally $S'' = F_4/Sq(3) \times Sq(1)$ with dim 28; it is Cartan´s FI space.

5 Magic Square and the E-Series: the Other Symmetric Spaces

For the E-series we anticipate our construction of the symmetric spaces is not so clearcut. One of the reasons is, undoubtedly, that our understanding of these groups is still incomplete, inspite the efforts of Freudenthal, Tits, and others. We shall resort to two contructions for these groups: (1), The *Magic Square* of Freudenthal, Rosenthal and Tits

(1956), and (2) an appeal to physics, in the existence of the chain of maximal supergravities, from spacetime dimensions 11 to $5, 4, 3$.

Recall the inclusions already seen $O(n) \subset U(n) \subset q(n) \subset U(2n) \subset O(4n)$, which holds for any n; it might be "filled in" for $n = 2$ and $n = 3$ as follows:

$$O(2) \ U(2) \ Sq(2) \ Oct(2) = \text{Spin}(9) \tag{15}$$

$$U(2) \ U(2)^2 \ U(4) \ Oct_C(2) = \text{Spin}(10) \tag{16}$$

$$Sq(2) \ U(4) \ O(8) \ Oct_H(2) = \text{Spin}(12) \tag{17}$$

$$\text{Spin}(9) \ \text{Spin}(10) \ \text{Spin}(12) \ \text{Spin}(16) = Oct_O(2) \tag{18}$$

Also, the primordial Magic Square is

$$O(3) \ U(3) \ Sq(3) \ Oct(2) = F_4 \tag{19}$$

$$U(3) \ U(3)^2 \ U(6) \ E_6 \tag{20}$$

$$Sq(3) \ U(6) \ O(12) \ E_7 \tag{21}$$

$$F_4 \ \ E_6 \ \ E_7 \ \ E_8 \tag{22}$$

The last row/column must be understood as $SOct(3) = F_4$, extended by the complex (E_6), the quaternions (E_7) and the octonions (E_8). Of course, the right way to understand these groups is as automorphism groups for the 3×3 hermitian Jordan algebras over the bi-rings octonions⊗reals = octonions, octonions⊗complex, octonions⊗quaternions and octonions⊗octonions. See e.g. [8]

Now for E_6: we have the three inclusions

$$F_4 \subset E_6, \quad U(6) \subset E_6, \quad O(10) \subset E_6 \tag{23}$$

which generates *three* of the 4 cases for E_6. For E_7 we have also the three inclusions

$$E_6 \subset E_7, \quad O(12) \subset E_7 \text{ (repeated!)} \tag{24}$$

which generates *two* of the 3 cases for E_7. For E_8 we have also two more cases:

$$E_7 \subset E_8, \quad O(16) \subset E_8 \text{ (repeated!)} \tag{25}$$

which generates the *two* cases for E_8.

The folowing eqs. summarizes the 3+2+2 cases:

$$E_6/F_4, \ E_6/(SU(6)\times SU(2)), \ E_6/SO(10)\times U(1) \ E_7/E_6\times U(1), \ E_7/O(12)\times Sq(1); \quad (26)$$

$$E_8/E_7\times Sq(1), \ E_8/O(16). \quad (27)$$

$$\text{with dimensions 26, 40, 32; 54, 64; 112, 128} \quad (28)$$

corresponding to Cartan´s EIV, EII, $EIII$; $EVII$, EVI; EIX and $EVIII$.

Before looking for the remaining two cases (one for each E_6, E_7) we pause to remark that the three following cases appear in physics:

$$F_4/\text{Spin}(9) = OP^2 \text{ (M Theory)}, \ E_6/\text{Spin}(10)\times SO(2) \text{ (F Theory)} \ E_8/SO(16) \text{ (Superstrings)} \quad (29)$$

The astute reader would have notices that the two remaining cases are the *split* form of the $E_{6,7}$ groups, namely the quotients by $Sq(4)$ and $SU(8)$ bzw., which correspond to the split form divided by the maximal compact subgroup of the mother group. Where are they? First, recall the notations

$$E_6^{compact} = E_{6(-78)}, \ E_6^{split} := E_{6(+6)} \approx R^{42}\times Sq(4) \text{ (dim 36)} \quad (30)$$

We have no better way to understand the remaining symmetric spaces that to draw the sequences of groups and subgroups appearing in Supergravity, from dim 11 down to dim 3, in such a way that the scalars form a sigma-like model. In fact, these pairs groups-subgroups are the *split* form:

$$E_8 \supset E_7 \supset E_6 \supset D_5 \supset A_4 \supset ... \quad (31)$$

$$O(16) \supset SU(8) \supset Sq(4) \supset Sq(2)^2 \supset Sq(2) \supset ... \quad (32)$$

which corresponds, in the descent from the 11-dim Supergravity of [7] to

$3d$, 128 *scalars*. $4d$, $35+35$ *scalars*. $5d$, 42 *scalars*. $6d$, 25 *scalars*, $7d$, 14 *scalars...*
$$(33)$$

It happens that the subgroup sequence corresponds exactly to the maximal compact subgroup of the main group sequence (I do not know why this is so).

This gives us *the remaining two* symmetric spaces: for E_6 we have: $Sq(4)$ with the traceless four-form makes up E_6, with $(36 + 42 = 78)$. For E_7 we have: $SU(8)$ with the four-form makes up E_7, as $63 + 70 = 133$. Finally, although we already obtained that, for E_8 we have: $SO(16)$ with the Spin irrep make up E_8, with $120+128 = 248$. In this connection see [9]. The first two were the missing exceptional symmetric spaces, corresponding

to Cartan´s EI, for $E_6/Sq(4)$, and EV for $E_7/SU(8)$.

Notice the sequences of groups $G(n)$, $G(n)^2$, $G(2n)$ and $O(n)$, $U(n)$, $Sq(n)$ appear repeated many times. To quote a single example, the chain of first Spin groups is $Spin(1) = O(1)$, $Spin(2) = U(1)$, $Spin(3) = Sq(1)$, $Spin(4) = Sq(1)^2 = q(1)$, $Spin(5) = Sq(2), Spin(6) = SU(4)$, etc. It reminds one of Bott´s periodicity.

6 Conclusions

1) The *twelve* exceptional symmetric spaces are obtained in a natural way, from pairs group/subgroup, both related to the octonions!
2) For G_2 and F_4, the arguments are unequivocal and unavoidable.
3) For the E series we present two arguments: one is the natural generalization of the inclusions $O \subset U \subset Sq \subset Oct$, but the second argument makes use of a series of E- groups and subgroups for which the evidence is physical, not mathematical! This undoubtedly reflects the fact, already alluded to, that our understanding of the E series is still incomplete.
4) During the writeup of our talk, the paper by Ferrara appeared [10], which exploit the relation among *irreducible Riemannian globally symmetric spaces* and *supergravity theories* in 3, 4 and 5 space-time dimensions, and therefore there is some overlap with our considerations on the last two exceptional symmetric spaces.

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